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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASS OF BOUNDED FUNCTIONS OF COMPLEX ORDER

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Abstract

In this paper, we obtain sharp coefficient bounds for functions analytic in the unit disc U and belonging to the class $R(\alpha, b, M)$, $b \neq 0$ is a complex number, $\alpha \geq 0$.

1. Introduction

Let A denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the unit disc U. Also denote by S the subclass of A, consisting of all univalent functions in U. Let Ω denote the class of bounded analytic functions w in U satisfying the conditions $w(0) = 0$ and $|w(z)| \leq |z|$ for $z \in U$. For $f \in A$, we say that f belongs to the class $F(b, M)(b \neq 0,$ complex, $M > 1/2)$, of bounded starlike functions of complex order, if and only if $\frac{f(z)}{z} \neq 0$ in U and for fixed M,

$$\left| \frac{b - 1 + \frac{zf'(z)}{f(z)}}{b} - M \right| < M, \quad z \in U. \tag{1.2}$$

The class $F(b, M)$ was studied by Nasr and Aouf [6].

In the present paper, we consider the class $R(\alpha, b, M)$ of functions $f \in A,$ satisfying the condition

$$\left| \frac{b - 1 + \frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}}}{b} - M \right| < M, \quad (M > 1/2;\, z \in U,\, \alpha \geq 0), \tag{1.3}$$

where $b \neq 0$, complex. We note that $R(1, 1, \infty) = R(1) = R$ [5] and $R(1, 1 - \alpha, \infty) = R_\alpha (0 \leq \alpha < 1)$ (Ahuja [1]).

We can easily show that $f \in R(a, b, M)$, if and only if there exists a function $w \in \Omega$ such that [3]

$$1 + \frac{1}{b}\left[\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1\right] = \frac{1 + w(z)}{1 - mw(z)}, \quad m = 1 - \frac{1}{M}, \, \alpha \geq 0. \tag{1.4}$$

Thus, from (1.4), it follows that $f \in R(\alpha, b, M)$ if and only for $z \in U$

$$\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} = \frac{1 + [(1 + m)b - m]w(z)}{1 - mw(z)}, \quad w(z) \in \Omega,\, m = 1 - \frac{1}{M}, \, \alpha \geq 0. \tag{1.5}$$

We shall need the following lemmas in our investigation:

Lemma 1.1 ([7]). *Let the function* w *defined by*

$$w(z) = \sum_{k=1}^{\infty} c_k z^k, \tag{1.6}$$

be in the class Ω. *Then* $|c_1| \leq 1$ *and* $|c_2| \leq 1 - |c_1|^2$.

Lemma 1.2 ([2]). *Let the function* w *defined by* (1.6) *be in the class* Ω. *Then*

$$|c_2 - \mu c_1^2| \leq \max\{1, |\mu|\}, \tag{1.7}$$

for any complex number μ. *Equality in* (1.7) *may be attained with the functions* $w(z) = z^2$ *and* $w(z) = z$ *for* $|\mu| < 1$ *and* $|\mu| \geq 1$, *respectively*.

Lemma 1.3 ([4]). *If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is an analytic function with positive real part in* Δ, *then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & \text{if } v \leq 0, \\ 2, & \text{if } 0 \leq v \leq 1, \\ 4v - 2, & \text{if } v \geq 1, \end{cases}$$

when $v < 0$ *or* $v > 1$, *the equality holds if and only if* $p_1(z)$ *is* $(1 + z)/(1 - z)$ *or one of its rotations. If* $0 < v < 1$, *then equality holds if and only if* $p_1(z)$ *is* $(1 + z^2)/(1 - z^2)$ *or one of its rotations. If* $v = 0$, *then equality holds if and only if*

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\gamma\right)\frac{1 + z}{1 - z} + \left(\frac{1}{2} - \frac{1}{2}\gamma\right)\frac{1 - z}{1 + z}, \quad (0 \leq \gamma \leq 1),$$

or one of its rotations. If $v = 1$, *then equality holds if and only if* $p_1(z)$ *is the reciprocal of one of functions such that the equality holds in the case of* $v = 0$. *Also, the above upper bound is sharp and it can be improved as follows when* $0 < v < 1$:

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \quad (0 < v \leq 1/2),$$

and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2, \quad (1/2 < v \leq 1).$$

2. Main Result

Theorem 2.1. *Let the function* f *defined by* (1.1) *be in the class* $R(\alpha, b, M)$. *Then*

(a) *for any complex number* μ, *we have*

$$|a_3 - \mu a_2^2| \leq \frac{(1+m)|b|}{(2+\alpha)} \max\left\{1, \frac{|2(1+\alpha)^2 m - (2+\alpha)(2\mu - 1 + \alpha)(1+m)b|}{2(1+\alpha)^2}\right\}. \tag{2.1}$$

(b)

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2}{(1+\alpha)^2}[(1+\alpha)^2(1+2m) - (2+\alpha)(2\mu - 1 + \alpha)(1+m)b], & \text{if } \mu \leq \sigma_1, \\ \dfrac{2(1+m)|b|}{2+\alpha}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\dfrac{2}{(1+\alpha)^2}[(1+\alpha)^2(1+2m) - (2+\alpha)(2\mu - 1 + \alpha)(1+m)b], & \text{if } \mu \geq \sigma_2, \end{cases} \tag{2.2}$$

where

$$\sigma_1 := \frac{2(1+\alpha)^2 m + (1-\alpha)(2+\alpha)(1+m)b}{2(2+\alpha)(1+m)b},$$

and

$$\sigma_2 := \frac{2(1+\alpha)^2 + (1-\alpha)(2+\alpha)b}{2(2+\alpha)b}.$$

The result is sharp.

Proof. Since $f \in R(\alpha, b, M)$, we have

$$\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} = \frac{1 + [(1+m)b - m]w(z)}{1 - mw(z)}, \quad \left(m = 1 - \frac{1}{M}, \alpha \geq 0\right), \tag{2.3}$$

where $w(z) = \sum_{k=1}^{\infty} c_k z^k \in \Omega$. From (2.3), we have

$$w(z) = \frac{\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1}{m\left[\frac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1\right] + (1+m)b}, \tag{2.4}$$

and then comparing the coefficients of z and z^2 on both sides of (2.4), we have

$$c_1 = \frac{(1+\alpha)a_2}{(1+m)b} \quad \text{and} \quad c_2 = \frac{(2+\alpha)a_3}{(1+m)b} - \left[m + \frac{(1-\alpha)(2+\alpha)(1+m)b}{2(1+\alpha)^2}\right]c_1^2.$$

Thus,

$$a_2 = \frac{(1+m)bc_1}{1+\alpha} \text{ and } a_3 = \frac{(1+m)b}{2+\alpha}\left[c_2 + \left(m + \frac{(1-\alpha)(2+\alpha)(1+m)b}{2(1+\alpha)^2}\right)c_1^2\right].$$

Hence

$$a_3 - \mu a_2^2 = \frac{(1+m)b}{2+\alpha}\left[c_2 - \frac{(2+\alpha)(2\mu - 1 + \alpha)(1+m)b - 2(1+\alpha)^2 m}{2(1+\alpha)^2} c_1^2\right],$$

and therefore,

$$|a_3 - \mu a_2^2| \leq \frac{(1+m)|b|}{2+\alpha}\left|c_2 - \frac{(2+\alpha)(2\mu - 1 + \alpha)(1+m)b - 2(1+\alpha)^2 m}{2(1+\alpha)^2} c_1^2\right|. \tag{2.5}$$

When μ is a complex number, applying Lemmas 1.1 and 1.2 in (2.5), we get (2.1) in Theorem 2.1(a).

Our results now follows by applying Lemma 1.2 in (2.5). The result is sharp for the function defined by

$$z^2 = \frac{\dfrac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1}{m\left[\dfrac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1\right] + (1+m)b},$$

when $\left|\dfrac{2(1+\alpha)^2 m - (2+\alpha)(2\mu - 1 + \alpha)(1+m)b}{2(1+\alpha)^2}\right| < 1,$

and

$$z = \frac{\dfrac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1}{m\left[\dfrac{z^{1-\alpha} f'(z)}{[f(z)]^{1-\alpha}} - 1\right] + (1+m)b},$$

when $\left|\dfrac{2(1+\alpha)^2 m - (2+\alpha)(2\mu - 1 + \alpha)(1+m)b}{2(1+\alpha)^2}\right| \geq 1.$

Theorem 2.1(b) now follows by an application of Lemma 1.3 in (2.5). To show that these bounds are sharp, we define the functions K_δ^ϕ $(\delta = 2, 3, \ldots)$ by

$$\frac{z^{1-\alpha}[K_\delta^\phi(z)]'}{[K_\delta^\phi(z)]^{1-\alpha}} = \frac{[1 + (1+m)b - m]z^{\delta-1}}{1 - mz^{\delta-1}}, \quad K_\delta^\phi(0) = 0 = (K_\delta^\phi(0))' - 1,$$

and the function F_γ and $G_\gamma (0 \leq \gamma \leq 1)$ by

$$\frac{z^{1-\alpha}[F_\gamma(z)]'}{[F_\gamma(z)]^{1-\alpha}} = \frac{[1 + (1+m)b - m]\left(\dfrac{z(z+\gamma)}{1+\gamma z}\right)}{1 - m\left(\dfrac{z(z+\gamma)}{1+\gamma z}\right)}, \quad F_\gamma(0) = 0 = (F_\gamma(0))' - 1,$$

and

$$\frac{z^{1-\alpha}[G_\gamma(z)]'}{[G_\gamma(z)]^{1-\alpha}} = \frac{[1+(1+m)b-m]\left(\frac{-z(z+\gamma)}{1+\gamma z}\right)}{1-m\left(\frac{-z(z+\gamma)}{1+\gamma z}\right)}, \quad G_\gamma(0) = 0 = (G_\gamma(0))' - 1.$$

Obviously, the functions K_δ^ϕ, F_γ, $G_\gamma \in R(\alpha, b, M)$. Also we write $K^\phi := K_2^\phi$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K^ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then equality holds if and only if f is K_3^ϕ or one of its rotations. If $\mu = \sigma_1$, then equality holds if and only if f is F_γ or one of its rotations. If $\mu = \sigma_2$, then equality holds if and only if f is G_γ or one of its rotations.

If $\sigma_1 \le \mu \le \sigma_2$, in view of Lemma 1.3, Theorem 2.1(b) can be improved. Let $f(z)$ given by (1.1) belongs to $R(\alpha, b, M)$ and σ_3 is given by

$$\sigma_3 := \frac{(1+\alpha)^2(1+2m)+(1-\alpha)(2+\alpha)(1+m)b}{2(2+\alpha)(1+m)b}.$$

If $\sigma_1 < \mu \le \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(2+\alpha)(2\mu-1+\alpha)(1+m)b - 2(1+\alpha)^2 m}{2(2+\alpha)(1+m)b}|a_2|^2 \le \frac{2(1+m)b}{2+\alpha}.$$

If $\sigma_3 < \mu \le \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{2(1+\alpha)^2 - (2+\alpha)(2\mu-1+\alpha)b}{2(2+\alpha)b}|a_2|^2 \le \frac{2(1+m)b}{2+\alpha}.$$

□

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